

Network growth with preferential attachment and without “rich get richer” mechanism

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Abstract

We propose a simple preferential attachment model of growing network using the complementary probability of Barabási-Albert (BA) model, i.e., $\Pi(k_i) \propto 1 - \frac{k_i}{\sum_j k_j}$. In this network, new nodes are preferentially attached to not well connected nodes. Numerical simulations, in perfect agreement with the master equation solution, give an exponential degree distribution. This suggests that the power law degree distribution is a consequence of preferential attachment probability together with “rich get richer” phenomena. We also calculate the average degree of a target node at time t ($< k_s(t) >$) and its fluctuations, to have a better view of the microscopic evolution of the network, and we also compare the results with BA model.

1 Introduction

In recent years there is a growing interest to study the evolution of complex networks and to develop models that reflect certain properties of the real networks using some statistical mechanics techniques, graph theory and computer simulations [1, 2, 3, 4]. One of the most important properties studied in networks is the degree distribution of nodes which is the probability $P(k)$ of a node to have degree k . We can distinguish three main laws of degree distribution: Poisson law where $P(k) = e^{-\langle k \rangle} \frac{\langle k \rangle^k}{k!}$, power law with $P(k) \sim k^{-\gamma}$ and γ represents the degree exponent, and exponential law with $P(k) \sim e^{-\frac{k}{c}}$ where c is constant. It appears that in nature most networks follow the last two distribution laws referred to above. Barabási-Albert reinvented price’s power law degree distribution network by introducing a simplified model based on both growth and preferential attachment. The resulting scale-free network is widely observed in variety of systems such as publication citation networks, many social networks, protein and gene networks. However, there are other real networks that follow an exponential law, for example, Worldwide Marine Transportation Network [5], the North American Power Grid Network [6], neural network of the *C.elegans* [7], and the Email Network at the University of Rovira i Virgili (ENURV) in Spain [8].

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The exponential law seems to be the result of growing network by randomly adding new nodes and links. On the other hand the power law seems to appear when nodes are added to the network one at time and are linked with nodes already well connected.

Many ideas on the formation of networks have been reviewed in the recent years. For example, Barabási, in his earlier work, asserted that the preferential attachment and growth are both required to generate a scale-free network [9]. Actually, it seems that growth is not necessary for such a purpose [10]. Furthermore, it is intuitive that preferential attachment without rich get richer effect does not generate a scale-free network [11]. Krapivsky et al. [12] have studied non linear preferential attachment with $\Pi(k_i) \propto k_i^\gamma$, and they shown that for $\gamma < 1$ the mechanism produces a stretched exponential degree distribution.

Despite many efforts, consistent theory of networks in evolution is still lacking and there is not yet a general principle predicting the topology of a formed network.

Aiming to understand the formation and the evolution of complex networks, many models were introduced to investigate the microscopic processes implicated in the resulting network structure.

In this context we introduce a simple complex network model growing with linear preferential attachment mechanism and without rich get richer effect. The objective is twofold: first to check if the power law degree distribution remain in the absence of the rich get richer scenario and, second, to see for eventual microscopic differences between scale-free and homogeneous networks.

2 Degree distribution

Similarly to the original BA model, our network evolves according to two mechanisms: the growth and the preferential attachment. Nodes entering the network prefer to attach to nodes with low degree, then the probability $\Pi(k_i)$ that one of the links of a new node connects to node i depends on its degree k_i such that $\Pi(k_i) = C(1 - \frac{k_i}{\sum_j k_j})$, where C is a normalization constant.

In connection with social networks, if we consider the degree of nodes as describing the wealth of people in a capitalist society, it is known [13] that we live in a world where rich get richer, but what kind of society we will have if there is no favors to rich people, and there is instead a continuous subvention to poor people?.

To implement our idea, we start with m_0 nodes, each one with m links. At every time step we add a new node with m edges that link the new node to m different nodes already present in the network. The probability that the new node is connected to a node i of degree k_i is $\Pi(k_i) = C(1 - \frac{k_i}{\sum_j k_j})$. The nor-

malization constant C is deduced from the condition $\sum_{i=1}^t \Pi(k_i) = 1$, which gives

$C = \frac{1}{t + m_0 - 1}$. t is the time when the last node was created and represents also the number of nodes added to the network.

For this model, the master equation can be written as:

$$(t+1)P(k, t+1) = tP(k, t) + m\Pi(k-1, t)tP(k-1, t) - m\Pi(k, t)tP(k, t) + \delta_{k,m}, \quad (1)$$

where δ is the Kronecker symbol.

The corresponding stationary equation takes the form:

$$(t+1)P(k) = tP(k) + m\left(1 - \frac{k-1}{2mt+mm_0}\right)\frac{tP(k-1)}{t-1} - m\left(1 - \frac{k}{2mt+mm_0}\right)\frac{tP(k)}{t-1} + \delta_{k,m}, \quad (2)$$

where we used $\sum_j k_j = 2mt + mm_0$. For large time we get

$$P(k) = \begin{cases} \frac{2mt - (k-1)}{2t + 2mt - k} P(k-1), & \text{for } k > m, \\ \frac{2t}{2t + 2mt - m}, & \text{for } k = m. \end{cases} \quad (3)$$

The above recurrence relation yields the following solution:

$$P(k) = \begin{cases} \frac{2t}{2t + 2mt - m} \prod_{j=m+1}^k \left(\frac{2mt - j + 1}{2t + 2mt - j} \right), & \text{for } k > m, \\ \frac{2t}{2t + 2mt - m}, & \text{for } k = m. \end{cases} \quad (4)$$

Although this equation is not in a closed form, numerical estimation of $P(k)$ is straightforward as shown in Fig. 1.

We also simulate the network with sizes up to $n = 2 \times 10^6$, initial number of nodes $m_0 = 3$ and $m = 2$. The simulation results strongly support the analytical findings (see Fig. 1).

We observed in simulations that k remains less than 40 for $t = 2 \cdot 10^6$, we then take $t \gg j$ in Eq. (4) and we obtain

$$P(k) \approx \begin{cases} \frac{1}{1+m} \left(\frac{m}{1+m} \right)^{k-m-1}, & \text{for } k > m, \\ \frac{1}{1+m}, & \text{for } k = m. \end{cases} \quad (5)$$

After normalization we get the exponential degree distribution $P(k) = Ae^{-A(k-m)}$, with $A = \ln\left(\frac{m+1}{m}\right)$. The inset in Fig. 1 shows the exponential form of $P(k)$ and the excellent agreement between simulations and theoretical results. This clearly confirms that the preferential attachment alone is not sufficient to produce scale-free networks.

3 Comparison with BA model

We search for differences between heterogeneous and homogeneous networks by comparing our model with the BA model. The degree distribution alone is not enough to characterize networks. Computing others microscopic quantities may help to have better insight into their evolution and formation. It turns out that scale-free network has nodes with important degree (hubs), while random network has no apparent structure. Evaluating the instantaneous average degree

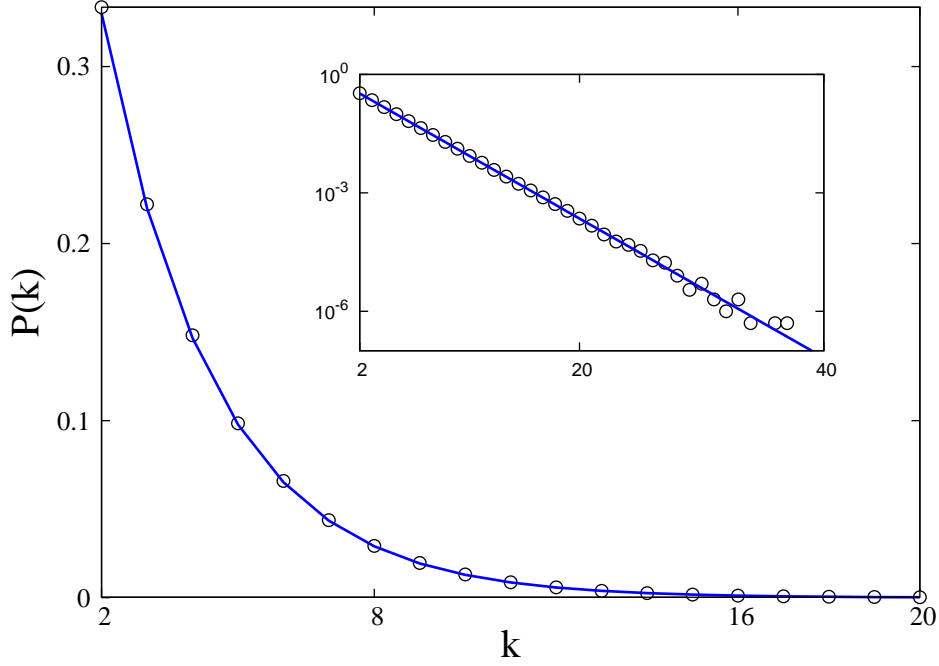


Figure 1: Simulation results (circles) for $n = 2 \cdot 10^6$, $m = 2$, $m_0 = 3$, and numerical solution (solid line) of Eq. (4). In the inset we plot the same data in the log-linear scale.

of target node $\langle k_s(t) \rangle$ and its fluctuations, can give quantitative information about hubs in the network. In fact, $\langle k_s(t) \rangle$ is somehow related to the instantaneous average degree of hubs, because when choosing randomly nodes, hubs have more chance to be selected.

Firstly, we analyze $\langle k_s(t) \rangle$ and $\langle k_s^2(t) \rangle$ in the BA network

$$\langle k_s(t) \rangle = \sum_{t_i=1}^t \Pi(k_i) k_i(t) + m_0 \Pi(k_0) k_0(t), \quad (6)$$

where $\Pi(k_i) = \frac{k_i(t)}{2mt + mm_0}$, t_i is the time when the node i was created, and $k_0(t)$ is the degree of initial nodes at time t .

Solving the mean field equation $\frac{\partial k_i(t)}{\partial t} = m \Pi(k_i)$, we obtain $k_i(t) = m \left(\frac{2t + m_0}{2t_i + m_0} \right)^{\frac{1}{2}}$.

Inserting the last expression in Eq. (6), we get

$$\langle k_s(t) \rangle = m \left(\sum_{t_i=1}^t \frac{1}{2t_i + m_0} + 1 \right) \quad (7)$$

$$= m \left(\ln(2t + m_0) + \gamma - a + \frac{1}{2(2t + m_0)} + O\left(\frac{1}{t^2}\right) \right), \quad (8)$$

where γ is the Euler constant, and $a = \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{1 + m_0}$.

Good agreement is obtained as shown in Fig. 2(a) between Eq. (8) and simulation results even for the first moments of the evolution. $\langle k_s(t) \rangle$ grows indefinitely with time and diverges for infinite network (or $t \rightarrow \infty$) due to the fact that, in heterogeneous networks, hubs are more likely to be selected and linked with new nodes.

On the other side, the average degree of the network remains finite [14, 15] since the majority of nodes have a small degree and the weight of hubs is small.

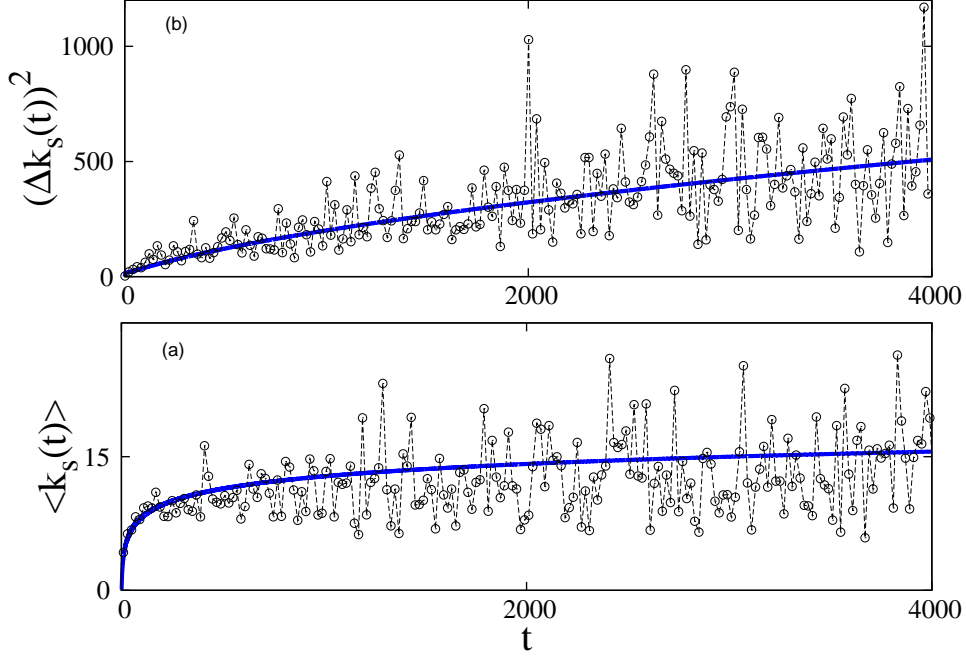


Figure 2: (a) Evolution of $\langle k_s(t) \rangle$ in the BA model, the solid line represents Eq. (8). (b) Evolution of fluctuations of $\langle k_s(t) \rangle$, the solid line represents Eq. (11). Circles joined by dashed lines in both cases are simulations data averaged over 20 runs for $m = 2$, $m_0 = 3$.

The second moment $\langle k_s^2(t) \rangle$ is written as

$$\langle k_s^2(t) \rangle = \sum_{t_i=1}^t \Pi(k_i) k_i^2(t) + m_0 \Pi(k_0) k_0^2(t) \quad (9)$$

$$\approx m^2 (2t + m_0)^{\frac{1}{2}} \left(\sum_{t_i=1}^t \frac{1}{(2t_i + m_0)^{\frac{3}{2}}} + m_0^{-\frac{1}{2}} \right). \quad (10)$$

For large time, $\sum_{t_i=1}^t \left(\frac{1}{t_i} \right)^{\frac{3}{2}} = \zeta\left(\frac{3}{2}\right) \approx 2.612$, we obtain $\langle k_s^2(t) \rangle \approx m^2 \sqrt{2t} (m_0^{-\frac{1}{2}} + 2.612 - b)$ with $b = 1 + \frac{1}{2^{\frac{3}{2}}} + \frac{1}{3^{\frac{3}{2}}} + \dots + \frac{1}{(1 + m_0)^{\frac{3}{2}}}$.

Fluctuations of $\langle k_s(t) \rangle$ are given by

$$(\Delta k_s(t))^2 \equiv \langle k_s(t)^2 \rangle - \langle k_s(t) \rangle^2 \approx m^2 \left[(m_0^{-\frac{1}{2}} + 2.612 - b) \sqrt{2t} - (\ln(2t))^2 \right], \quad (11)$$

which become arbitrary large when time increases sufficiently.

Simulation data, in accordance with Eq. (11) (see Fig. 2(b)), shows the increasing tendency of fluctuations in $\langle k_s(t) \rangle$. This can be explained by the fact that the maximum degree in the network $k_{max} \sim \sqrt{t}$ increases [15] faster than $\langle k_s(t) \rangle \sim \ln(t)$ (Eq. (8)) and the difference between the two quantities becomes greater with time.

We now turn to the same analysis in our model. The mean field evolution equation for $k_i(t)$ gives

$$\frac{\partial k_i(t)}{\partial t} + \frac{k_i(t)}{(2t + m_0)(t + m_0 - 1)} = \frac{m}{t + m_0 - 1}. \quad (12)$$

The solution has the form

$$k_i(t) = m \left(\frac{t + m_0 - 1}{2t + m_0} \right)^{\frac{1}{m_0 - 2}} \left[\left(\frac{t_i + m_0 - 1}{2t_i + m_0} \right)^{-\frac{1}{m_0 - 2}} - A(t_i) + A(t) \right], \quad (13)$$

where $A(t) = \int_1^t \frac{\left(\frac{t' + m_0 - 1}{2t' + m_0} \right)^{-\frac{1}{m_0 - 2}}}{t' + m_0 - 1} dt'$.

The average value of target node $\langle k_s(t) \rangle$ is obtained immediately for any time t by substituting Eq. (13) into Eq. (6). The resulting equation is solved numerically as shown in Fig. 3(a).

For large time and taking $t \gg m_0$, we find $A(t) \approx 2^{\frac{1}{m_0 - 2}} \ln(t)$, $k_i(t) \approx m \left(1 + \ln \frac{t}{t_i} \right)$, then

$$\begin{aligned} \langle k_s(t) \rangle &\approx \frac{m}{t} \left(\sum_{t_i=1}^t \ln(t) - \ln(t_i) + 1 \right) \\ &\approx \frac{m}{t} \left(t \left(\ln(t) + 1 \right) - \left(\sum_{t_i=1}^t \ln(t_i) \right) \right) \\ &\approx \frac{m}{t} \left(t \left(\ln(t) + 1 \right) - \ln(t_i!) \right) \\ &\approx 2m \end{aligned} \quad (14)$$

This is the value of the node average connectivity of a network growing with uniform attachment resulting in an exponential distribution of degree [9].

The second moment is obtained by substituting the corresponding expressions of $\Pi(k_i)$ and $k_i(t)$ in Eq. (9), we get for large time

$$\langle k_s^2(t) \rangle \approx \frac{m^2}{t} \left(\sum_{t_i=1}^t \left(\ln\left(\frac{t}{t_i}\right) + 1 \right)^2 \right). \quad (15)$$

Making the approximations $\sum_{t_i=1}^t \ln(t_i) \approx t \ln(t) - t$, and $\sum_{t_i=1}^t \ln(t_i)^2 \approx t \ln(t)^2 - 2t \ln(t) + 2t - 2$, we find $\langle k_s^2(t) \rangle \approx 5m^2$.

Fluctuations are $(\Delta k_s(t))^2 \equiv \langle k_s^2(t) \rangle - \langle k_s(t) \rangle^2 \approx m^2$. This finding, together with $\langle k_s(t) \rangle \approx 2m$, show that almost all nodes have the same degree as illustrated in Fig. 3(b). The homogeneity of the network can be explained by the fact that the preferential attachment used here doesn't allow the formation of hubs, since it neither allows the rich to get richer, nor it enriches the poor.

4 Conclusion

In this work, we have introduced a simple model of complex network with a preferential attachment criteria and without "rich get richer" effect. The network obtained is homogeneous, which demonstrates the crucial role of the "rich

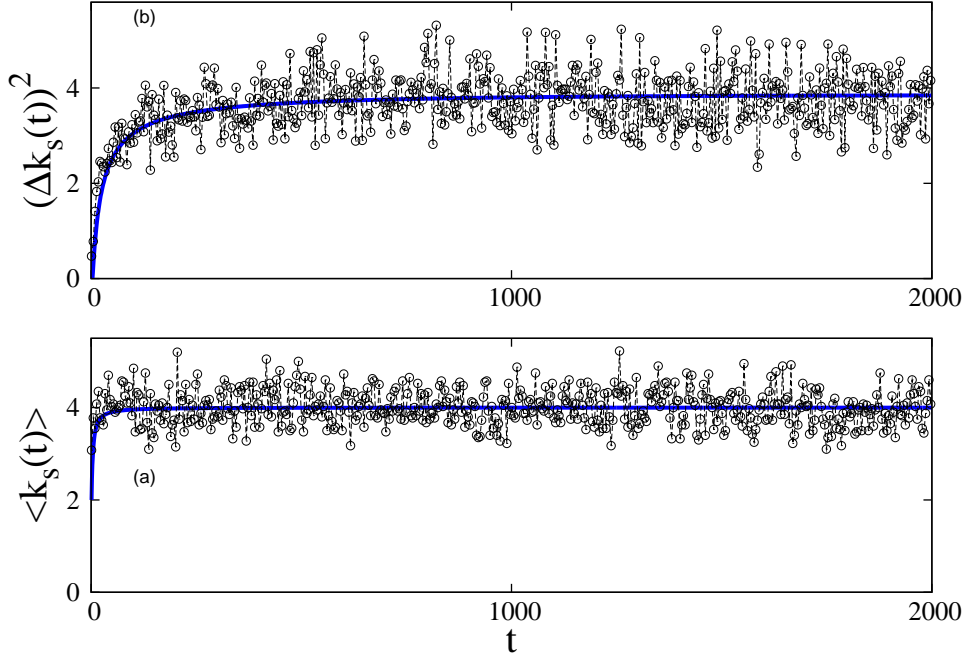


Figure 3: (a) Evolution of $\langle k_s(t) \rangle$ in our model, the solid line represents Eq. (14). (b) Evolution of fluctuations of $\langle k_s(t) \rangle$, the solid line represents the numerical solution of Eq. (14) and Eq. (15). Circles joined by dashed lines in both cases are simulations data averaged over 20 runs for $m = 2$, $m_0 = 3$.

get richer” in the topology of the network. Giving preferential treatment to the least connected nodes is equivalent to use a random attachment probability. In terms of social wealth distribution, Pareto principle [13] doesn’t apply and we have instead an exponential distribution of income.

Computing the instantaneous average degree of a target node and its fluctuations provide more information than the usual average degree of the network, in particular we show how the average degree of hubs and its fluctuations diverge with time in the BA model, and stay finite in our model.

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